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Singular Differential Equations with Linear and Nonlinear Boundary Conditions

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Abstract—Existence results are established for the singular equation $y'' + f(t,y) = 0$, where f is not a Carathéodory function. Our nonlinearity f is allowed to change sign.

Keywords—Singular, Positive, Nonlinear, Boundary value problems.

1. INTRODUCTION

This paper discusses the second-order differential equation $y'' = f(t,y)$. Here f is not a Carathéodory function due to the singular behavior of its y variable and also the singular behavior of its t variable. Many physical situations are modelled by problems of this kind, for example, problems in gas and fluid dynamics [1,2].

In [3], Taliaferro showed that

$$\begin{gathered} y'' + q(t)y^{-\alpha} = 0, \qquad 0 < t < 1, \\ y(0) = 0 = y(1), \end{gathered}$$

has a $C[0,1] \cap C^2(0,1)$ solution; here $\alpha > 0$, $q \in C(0,1)$, $q > 0$ on $(0,1)$ and $\int_0^1 t(1-t)q(t)\,dt < \infty$. Since Taliaferro's paper, many authors [4–9] have examined the more general boundary value problem

$$\begin{gathered} y'' + f(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = 0 = y(1), \end{gathered} \tag{1.1}$$

where $f(t,y) > 0$, for $t \in (0,1)$ and $y > 0$. However, recently, Habets and Zanolin [10] have examined (1.1) where f is allowed to change sign. They show, using monotonicity methods, the following.

HYPOTHESIS (H1). *If there exists a constant $L > 0$ such that for any compact set $K \subset (0,1)$, there is an $\epsilon > 0$ with $f(t,x) > L$ for all $t \in K$, $x \in (0,\epsilon]$.*

HYPOTHESIS (H2). *If for any $\delta > 0$, there is a $h_\delta \in C(0,1)$ such that $|f(t,x)| \le h_\delta(t)$ for all $t \in (0,1)$, $x \ge \delta$; here $\int_0^1 s(1-s)h_\delta(s)\,ds < \infty$.*

If (H1) and (H2) are satisfied, then (1.1) has a nonnegative solution.

Our paper is divided into two main parts. In Section 2, we discuss the boundary value problem (1.1). We establish, using a technique initiated in [11], solutions to (1.1) which generalise and

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complement the results in [10]. In particular, we relax the Conditions (H1) and (H2). Several existence theorems are established. In Section 3, we examine the more general situation, namely,

$$\begin{aligned} &y'' + f(t,y) = 0, \qquad 0 < t < 1, \\ &y(0) = 0, \qquad \theta\left(y'(1)\right) + y(1) = 0, \end{aligned} \tag{1.2}$$

where θ may be nonlinear. Of course our results include the Sturm-Liouville condition $ky'(1) + y(1) = 0$, where $k \geq 0$. An existence result in the spirit of Section 2 is established for (1.2).

The analysis used in this paper rely on fixed-point methods. We state for convenience, the two fixed-point theorems we will use.

THEOREM 1.1. *(See [12].) Let K be a convex subset of a normed linear space E. Then every compact map $F : K \to K$ has at least one fixed point.*

THEOREM 1.2. *(See nonlinear alternative, [12,13].) Assume U is a relatively open subset of a convex set K in a normed linear space E. Let $N : \overline{U} \to K$ be a compact map with $p \in U$. Then either*

(i) *N has a fixed point in $\overline{U}$; or*
(ii) *there is a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda N u + (1-\lambda)p$.*

REMARK. By a map being *compact*, we mean it is continuous with relatively compact range. For later purposes, a map is *completely continuous*, if it is continuous and the image of every bounded set in the domain is contained in a compact set in the range.

2. DIRICHLET BOUNDARY CONDITION

Several existence results are established for the singular problem

$$\begin{aligned} y'' + f(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = y(1) = 0. \end{aligned} \tag{2.1}$$

These results rely on the following existence principle for the boundary value problem

$$\begin{aligned} y'' + g(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = a, \qquad y(1) = b, \end{aligned} \tag{2.2}$$

which was established, in a less general situation in [14]; for completeness we sketch the proof.

REMARK. By a solution to (2.2), we mean a function $y \in C[0,1] \cap C^2(0,1)$ which satisfies the differential equation on (0,1) and the stated boundary conditions.

THEOREM 2.1. *Suppose*

$$g : (0,1) \times \mathbf{R} \to \mathbf{R} \text{ is continuous.} \tag{2.3}$$

(i) *Assume*

for each $r > 0$ there exists a $h_r \in C(0,1)$ with $\int_0^1 t(1-t)h_r(t)\,dt < \infty$ such that $|y| \leq r$ implies $|g(t,y)| \leq h_r(t)$ for $t \in (0,1)$. Also, assume $\lim_{t\to 0^+} t^2(1-t)h_r(t) = 0$ if $\int_0^1 (1-x)h_r(x)\,dx = \infty$ and $\lim_{t\to 1^-} t(1-t)^2 h_r(t) = 0$ if $\int_0^1 x h_r(x)\,dx = \infty$ (2.4)

holds. In addition, suppose there is a constant M, independent of λ, with $|y|_0 = \sup_{[0,1]} |y(t)| \leq M$ for any solution y to

$$\begin{aligned} y'' + \lambda g(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = a, \qquad y(1) = b, \end{aligned} \tag{2.5$_\lambda$}$$

for each $\lambda \in (0,1)$. Then (2.2) has a solution.

(ii) *Assume*

$$\text{there exists a } h \in C(0,1) \text{ with } \int_0^1 t(1-t)h(t)\,dt < \infty \text{ such that } |g(t,y)| \le h(t) \text{ for } t \in (0,1) \text{ and } y \in \mathbf{R}. \text{ Also, assume } \lim_{t\to 0^+} t^2(1-t)h(t) = 0 \text{ if } \int_0^1 (1-x)h(x)\,dx = \infty \text{ and } \lim_{t\to 1^-} t(1-t)^2 h(t) = 0 \text{ if } \int_0^1 xh(x)\,dx = \infty \tag{2.6}$$

holds. Then (2.2) has a solution.

PROOF.

(i) Suppose $y \in C[0,1]$. Then $xg(x,y(x)) \in L^1[0,t]$ for $t < 1$, since $\int_0^t x|g(x,y(x))|\,dx \le 1/(1-t)\int_0^t x(1-x)|g(x,y(x))|\,dx$. Also,

$$\lim_{t\to 1^-} (1-t)\int_0^t sg(s,y(s))\,ds = 0,$$

from (2.4). Similarly, $(1-x)g(x,y(x)) \in L^1[t,1]$ for $t > 0$ and $\lim_{t\to 0^+} t\int_t^1 (1-s)g(s,y(s))\,ds = 0$. Consequently,

$$r(t) = a(1-t) + bt + \lambda(1-t)\int_0^t sg(s,y(s))\,ds + \lambda t\int_t^1 (1-s)g(s,y(s))\,ds \in C[0,1],$$

with $r(0) = a$ and $r(1) = b$.

It is now easy to check that solving $(2.5)_\lambda$ is equivalent to finding a solution $y \in C[0,1]$ to

$$y(t) = a(1-t) + bt + \lambda(1-t)\int_0^t sg(s,y(s))\,ds + \lambda t\int_t^1 (1-s)g(s,y(s))\,ds. \tag{2.7$_\lambda$}$$

Define the operator $N : C[0,1] \to C[0,1]$ by

$$Ny(t) = a(1-t) + bt + (1-t)\int_0^t sg(s,y(s))\,ds + t\int_t^1 (1-s)g(s,y(s))\,ds. \tag{2.8}$$

Then $(2.7)_\lambda$ is equivalent to the fixed-point problem

$$y = (1-\lambda)p + \lambda Ny, \qquad \text{where } p = a(1-t) + b.$$

We claim that $N : C[0,1] \to C[0,1]$ is continuous and completely continuous. Continuity follows immediately from the Lebesgue dominated convergence theorem and the fact that

$$\begin{aligned} |Ny_n(t) - Ny(t)| &\le (1-t)\int_0^t s\,|g(s,y_n(s)) - g(s,y(s))|\,ds \\ &\quad + t\int_t^1 (1-s)\,|g(s,y_n(s)) - g(s,y(s))|\,ds \\ &\le \int_0^1 s(1-s)\,|g(s,y_n(s)) - g(s,y(s))|\,ds, \end{aligned}$$

with $y_n, y \in C[0,1]$. To show complete continuity, we will use the Arzela-Ascoli Theorem. To see this let $\Omega \subseteq C[0,1]$ be bounded, i.e., there exists $M_0 > 0$ with $|u|_0 \le M_0$ for each $u \in \Omega$. Now if $u \in \Omega$, we have

$$|(Nu)'(t)| \le |a| + \int_0^t sh_{M_0}(s)\,ds + \int_t^1 (1-s)h_{M_0}(s)\,ds \equiv \tau_{M_0}(t), \tag{2.9}$$

where h_{M_0} is determined from the bounded set Ω and (2.4). It is easy to check that $\tau_{M_0} \in L^1[0,1]$. Hence, (2.8) and (2.9) imply that $N\Omega$ is a bounded equicontinuous family on $[0,1]$. Consequently, the Arzela-Ascoli Theorem implies $N : C[0,1] \to C[0,1]$ is completely continuous. Set

$$U = \{u \in C[0,1] : |u|_0 < \max\{M,p\} + 1\}, \qquad K = E = C[0,1].$$

Theorem 1.2 guarantees that N has a fixed point, i.e., $(2.7)_1$ has a solution $y \in C[0,1]$.

(ii) Solving (2.2) is equivalent to the fixed-point problem

$$y = Ny,$$

where N is as described in (2.8). Essentially the same reasoning as in Part (i) implies $N : C[0,1] \to C[0,1]$ is continuous and *compact* (since (2.6) holds). Theorem 1.1 guarantees that N has a fixed point. ∎

Our first existence result uses Theorem 2.1(i).

THEOREM 2.2. *Suppose the following conditions are satisfied:*

$$f : (0,1) \times (0,\infty) \to \mathbf{R} \text{ is continuous;} \tag{2.10}$$

$|f(t,y)| \le q_1(t)g(y) + q_2(t)h(y)$ on $(0,1) \times (0,\infty)$ with $g > 0$ continuous and nonincreasing on $(0,\infty)$, $h \ge 0$ continuous on $[0,\infty)$, and h/g nondecreasing on $(0,\infty)$; here $q_i \in C(0,1)$, $i = 1,2$, with $q_i > 0$ on $(0,1)$ and $\int_0^1 q_i(x)\,dx < \infty$; (2.11)

let $n \in \{3,4,\dots\}$ and associated with each n we have a constant ρ_n such that $\{\rho_n\}$ is a nonincreasing sequence with $\lim_{n\to\infty} \rho_n = 0$, and such that for $1/n \le t \le 1 - 1/n$ we have $f(t,\rho_n) \ge 0$; (2.12)

there exists a constant $M > 0$ such that for $z > 0$, $\int_0^z (du/g(u)) \le r_1 + r_2(h(z)/g(z)) + \int_0^{\rho_3}(du/g(u))$ implies $z \le M$; here $r_i = \max\{2\int_0^{1/2} t(1-t)q_i(t)\,dt,\ 2\int_{1/2}^1 t(1-t)q_i(t)\,dt\}$, $i = 1,2$; (2.13)

there exists a function $\alpha \in C[0,1] \cap C^2(0,1)$ with $\alpha(0) = \alpha(1) = 0$, $\alpha > 0$ on $(0,1)$ such that $f(t,y) + \alpha''(t) > 0$ for $(t,y) \in (0,1) \times \{y \in (0,\infty) : y < \alpha(t)\}$ (2.14)

and

for any $R > 0$, $(1/g)$ is differentiable on $(0,R]$ with $g' < 0$ a.e. and $g'/g^2 \in L^1[0,R]$; in addition $\int_0^\infty (|g'(t)|^{1/2})/(g(t))\,dt = \infty$. (2.15)

Then (2.1) has a solution in $C[0,1] \cap C^2(0,1)$.

PROOF. Fix $n \in \{3,4,\dots\}$. We begin by showing that

$$\begin{gathered} y'' + f^\star(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = y(1) = \rho_n, \end{gathered} \tag{2.16^n}$$

has a solution; here

$$f^\star(t,y) = \begin{cases} f(t,y), & y \ge \rho_n, \\ f(t,\rho_n) + \rho_n - y, & y < \rho_n \text{ and } \dfrac{1}{n} \le t \le 1 - \dfrac{1}{n}, \\ f\left(\dfrac{1}{n},\rho_n\right) + \rho_n - y, & y < \rho_n \text{ and } 0 \le t \le \dfrac{1}{n}, \\ f\left(1 - \dfrac{1}{n},\rho_n\right) + \rho_n - y, & y < \rho_n \text{ and } 1 - \dfrac{1}{n} \le t \le 1. \end{cases}$$

REMARK. Notice (2.10) and (2.11) imply that $g = f^\star$ satisfies (2.3) and (2.4).

To show $(2.16)^n$ has a solution, we consider the family of problems

$$\begin{aligned} y'' + \lambda f^\star(t,y) = 0, \qquad 0<t<1, \quad 0<\lambda<1, \\ y(0)=y(1)=\rho_n. \end{aligned} \tag{2.17$^n_\lambda$}$$

We first show that

$$y(t) \geq \rho_n, \qquad t \in [0,1], \tag{2.18}$$

for any solution $y \in C[0,1] \cap C^2(0,1)$ to $(2.17)^n_\lambda$. To see this, suppose $y - \rho_n$ has a negative minimum at $t_0 \in (0,1)$ in which case $y'(t_0) = 0$ and $y''(t_0) \geq 0$. However,

$$y''(t_0) = -\lambda f^\star(t_0, y(t_0)) = \begin{cases} -\lambda\left[f(t_0,\rho_n) + \rho_n - y(t_0)\right], & \text{if } \frac{1}{n} \leq t_0 \leq 1 - \frac{1}{n}, \\ -\lambda\left[f\left(\frac{1}{n},\rho_n\right) + \rho_n - y(t_0)\right], & \text{if } 0 \leq t_0 \leq \frac{1}{n}, \\ -\lambda\left[f\left(1-\frac{1}{n},\rho_n\right) + \rho_n - y(t_0)\right], & \text{if } 1-\frac{1}{n} \leq t_0 \leq 1, \end{cases}$$

i.e., $y''(t_0) < 0$, a contradiction. Thus, (2.18) holds. Now since $y(0) = y(1) = \rho_n$, we may assume the absolute maximum of y occurs at, say, $t_n \in (0,1)$, so $y'(t_n) = 0$. For $x \in (0,1)$, we have

$$\frac{-y''(x)}{g(y(x))} \leq q_1(x) + q_2(x)\frac{h(y(x))}{g(y(x))}. \tag{2.19}$$

Integrate from $t\,(t<t_n)$ to t_n, to obtain

$$\frac{y'(t)}{g(y(t))} + \int_t^{t_n}\left\{\frac{-g'(y(x))}{g^2(y(x))}\right\}[y'(x)]^2\,dx \leq \int_t^{t_n} q_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_t^{t_n} q_2(x)\,dx,$$

and so

$$\frac{y'(t)}{g(y(t))} \leq \int_t^{t_n} q_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_t^{t_n} q_2(x)\,dx.$$

Integrate from 0 to t_n, to obtain

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leq \int_0^{t_n} xq_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_0^{t_n} xq_2(x)\,dx + \int_0^{\rho_n}\frac{du}{g(u)},$$

and so

$$\begin{aligned} \int_0^{y(t_n)} \frac{du}{g(u)} \leq{}& \frac{1}{1-t_n}\int_0^{t_n} x(1-x)q_1(x)\,dx \\ &+ \frac{h(y(t_n))}{g(y(t_n))}\frac{1}{1-t_n}\int_0^{t_n} x(1-x)q_2(x)\,dx + \int_0^{\rho_3}\frac{du}{g(u)}. \end{aligned} \tag{2.20}$$

Similarly, if we integrate (2.19) from t_n to $t\,(t \geq t_n)$ and then from t_n to 1, we obtain

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leq \frac{1}{t_n}\int_{t_n}^1 x(1-x)q_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\frac{1}{t_n}\int_{t_n}^1 x(1-x)q_2(x)\,dx + \int_0^{\rho_3}\frac{du}{g(u)}. \tag{2.21}$$

Now (2.20) and (2.21) imply

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leq r_1 + r_2\frac{h(y(t_n))}{g(y(t_n))} + \int_0^{\rho_3}\frac{du}{g(u)},$$

where r_1, r_2 are as in (2.13). Consequently, (2.13) implies $y(t_n) \le M$, and so

$$\rho_n \le y(t) \le M, \qquad \text{for } t \in [0,1]. \tag{2.22}$$

Now, Theorem 2.1(i) implies $(2.16)^n$, which has a solution $y_n \in C[0,1] \cap C^2(0,1)$ (in fact, in $C^1[0,1] \cap C^2(0,1)$ since $\int_0^1 q_i(x)\,dx < \infty$) with

$$\rho_n \le y_n(t) \le M, \qquad \text{for } t \in [0,1]. \tag{2.23}$$

Also, y_n is a solution of

$$\begin{gathered} y'' + f(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = y(1) = \rho_n. \end{gathered}$$

Next we will obtain a sharper lower bound on y_n, namely, we will show

$$\alpha(t) \le y_n(t) \le M, \qquad \text{for } t \in [0,1], \tag{2.24}$$

where α is as in (2.14).

If (2.24) is not true then $y_n - \alpha$ would have a negative minimum at, say, $t_0 \in (0,1)$. In this case, $y_n''(t_0) - \alpha''(t_0) \ge 0$. However, since $0 < y_n(t_0) < \alpha(t_0)$ and $y_n(t_0) \ge \rho_n$, we have

$$y_n''(t_0) - \alpha''(t_0) = -\left[f\left(t_0, y_n(t_0)\right) + \alpha''(t_0)\right] < 0,$$

a contradiction. Hence (2.24) is true.

We shall now obtain a solution to (2.1) by means of the Arzela-Ascoli Theorem, as a limit of solutions of $(2.16)^n$. To this end, we will show

$$\{y_n\}_{n=3}^{\infty} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{2.25}$$

Of course, $\{y_n\}$ is uniformly bounded by (2.24). To show equicontinuity, some more estimates are needed.

We have

$$-y_n''(x) \le g(y_n(x)) \left\{ q_1(x) + q_2(x) \frac{h(M)}{g(M)} \right\}, \qquad \text{for } x \in (0,1). \tag{2.26}$$

Divide (2.26) by $g(y_n(x))$ and integrate from 0 to 1 to obtain

$$\frac{-y_n'(1)}{g(\rho_n)} + \frac{y_n'(0)}{g(\rho_n)} + \int_0^1 \left\{ \frac{-g'(y_n(x))}{g^2(y_n(x))} \right\} [y_n'(x)]^2\,dx \le \int_0^1 \left[q_1(x) + q_2(x) \frac{h(M)}{g(M)} \right] dx.$$

Then, since $y_n'(0) \ge 0$, $y_n'(1) \le 0$ (note, $y_n(0) = y_n(1) = \rho_n$ and $y_n \ge \rho_n$ on $[0,1]$), we have

$$\int_0^1 \left\{ \frac{-g'(y_n(x))}{g^2(y_n(x))} \right\} [y_n'(x)]^2\,dx \le \int_0^1 \left[q_1(x) + q_2(x) \frac{h(M)}{g(M)} \right] dx \equiv K_1. \tag{2.27}$$

Now consider

$$I(z) = \int_0^z \frac{[-g'(u)]^{1/2}}{g(u)}\,du.$$

Notice I is an increasing map from $[0,\infty)$ onto $[0,\infty)$, with I continuous on $[0,\Omega]$ for any $\Omega > 0$. For $t, s \in [0,1]$, we have from Hölder's inequality that

$$\begin{aligned} |I(y_n(t)) - I(y_n(s))| &= \left| \int_{y_n(s)}^{y_n(t)} \frac{[-g'(u)]^{1/2}}{g(u)}\,du \right| = \left| \int_s^t \frac{[-g'(y_n(x))]^{1/2}}{g(y_n(x))} y_n'(x)\,dx \right| \\ &\le |t-s|^{1/2} \left(\int_0^1 \left\{ \frac{-g'(y_n(x))}{g^2(y_n(x))} \right\} [y_n'(x)]^2\,dx \right)^{1/2} \le K_1^{1/2} |t-s|^{1/2}. \end{aligned}$$

It follows from this inequality, the uniform continuity of I^{-1} on $[0, I(M)]$ and

$$|y_n(t) - y_n(s)| = \left|I^{-1}\left(I\left(y_n(t)\right)\right) - I^{-1}\left(I\left(y_n(s)\right)\right)\right|,$$

that $\{y_n\}$ is equicontinuous on $[0,1]$. Thus (2.25) is established.

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N of integers and a function $y \in C[0,1]$ with y_n converging uniformly on $[0,1]$ to y as $n \to \infty$ through N. Also, $y(0) = y(1) = 0$ and $\alpha(t) \leq y(t) \leq M$ for $t \in [0,1]$. Now $y_n, n \in N$, satisfies the integral equation

$$y_n(t) = y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t) f(s, y_n(s))\, ds, \qquad \text{for } t \in (0,1). \tag{2.28}$$

Notice (2.28) (take $t = 3/4$ say) implies $\{y_n'(1/2)\}$, $n \in N$ is a bounded sequence since $\alpha(t) \leq y_n(t) \leq M$ for $t \in [0,1]$. Thus, $\{y_n'(1/2)\}$ has a convergent subsequence; for convenience let $\{y_n'(1/2)\}$, $n \in N$ denote this subsequence also, and let $r \in \mathbf{R}$ be its limit.

Fix $t \in (0,1)$. Since f is uniformly continuous on compact subsets of $[\min((1/2),t), \max((1/2), t)] \times (0, M]$, let $n \to \infty$ through N in (2.28) to obtain

$$y(t) = y\left(\frac{1}{2}\right) + r\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t) f(s, y(s))\, ds.$$

Thus, $y \in C^2(0,1)$ and $-y''(t) = f(t, y(t))$ for $t \in (0,1)$. ∎

Our next existence result uses Theorem 2.1(ii).

THEOREM 2.3. *Suppose (2.10)–(2.12), (2.14), and (2.15) hold. In addition, assume*

there exists a function $\beta \in C[0,1] \cap C^2(0,1)$ with $\beta \geq \rho_3$ on $[0,1]$, such that $f(t, \beta(t)) + \beta''(t) \leq 0$ for $t \in (0,1)$ (2.29)

is satisfied. Then (2.1) has a solution in $C[0,1] \cap C^2(0,1)$.

PROOF. Fix $n \in \{3, 4, \dots\}$. We first show

$$\begin{aligned} y'' + f(t,y) = 0, &\qquad 0 < t < 1, \\ y(0) = y(1) = \rho_n, & \end{aligned} \tag{2.30^n}$$

has a solution. The idea is to look at

$$\begin{aligned} y'' + f^{**}(t,y) = 0, &\qquad 0 < t < 1, \\ y(0) = y(1) = \rho_n, & \end{aligned} \tag{2.31^n}$$

where

$$f^{**}(t,y) = \begin{cases} f\left(t, \beta(t)\right) + r\left(\beta(t) - y\right), & y > \beta(t), \\ f(t,y), & \rho_n \leq y \leq \beta(t), \\ f(t, \rho_n) + r(\rho_n - y), & y < \rho_n \text{ and } \dfrac{1}{n} \leq t \leq 1 - \dfrac{1}{n}, \\ f\left(\dfrac{1}{n}, \rho_n\right) + r(\rho_n - y), & y < \rho_n \text{ and } 0 \leq t \leq \dfrac{1}{n}, \\ f\left(1 - \dfrac{1}{n}, \rho_n\right) + r(\rho_n - y), & y < \rho_n \text{ and } 1 - \dfrac{1}{n} \leq t \leq 1, \end{cases}$$

and $r : \mathbf{R} \to [-1, 1]$ is the radial retraction defined by

$$r(u) = \begin{cases} u, & \text{if } |u| \leq 1, \\ \dfrac{u}{|u|}, & \text{otherwise.} \end{cases}$$

REMARK. Notice $g = f^{**}$ satisfies (2.3) and (2.6).

Now Theorem 2.1(ii) implies $(2.31)^n$ has a solution $y_n \in C[0,1] \cap C^2(0,1)$. Essentially, the same reasoning as in Theorem 2.2 yields

$$y_n(t) \geq \rho_n, \qquad \text{for } t \in [0,1]. \tag{2.32}$$

Next we claim

$$y_n(t) \leq \beta(t), \qquad \text{for } t \in [0,1]. \tag{2.33}$$

If (2.33) is not true, then $y_n - \beta$ would have a positive maximum at, say, $t_0 \in (0,1)$ in which case $y_n''(t_0) - \beta''(t_0) \leq 0$. However, since $y_n(t_0) > \beta(t_0)$, we have

$$y_n''(t_0) - \beta''(t_0) = -\left[f\left(t_0, \beta(t_0)\right) + r\left(\beta(t_0) - y_n(t_0)\right) + \beta''(t_0)\right] > 0,$$

a contradiction. Thus, $\rho_n \leq y_n(t) \leq \beta(t)$ for $t \in [0,1]$, and so y_n is a solution of $(2.30)^n$. Essentially, the same reasoning as in Theorem 2.2 (from equation (2.24) onwards) establishes that (2.1) has a solution. ∎

We now obtain a "general upper and lower solution theorem" for singular problems.

THEOREM 2.4. *Suppose (2.10), (2.12), and (2.14) hold. Assume,*

> *$|f(t,y)| \leq q_1(t)g(y) + q_2(t)h(y)$ on $(0,1) \times (0,\infty)$ with $g > 0$ continuous and nonincreasing on $(0,\infty)$ and $h \geq 0$ continuous on $[0,\infty)$; here $q_i \in C(0,1)$, $i = 1,2$ with $q_i > 0$ on $(0,1)$ and $\int_0^1 x(1-x)q_i(x)\,dx < \infty$. Also, assume $\lim_{t\to 0^+} t^2(1-t)q_i(t) = 0$ if $\int_0^1 (1-x)q_i(x)\,dx = \infty$ and $\lim_{t\to 1^-} t(1-t)^2 q_i(t) = 0$ if $\int_0^1 x q_i(x)\,dx = \infty$, $i = 1,2$;* (2.34)

and

> *for each $n \in \{3,4,\dots\}$, there exists a function $\beta_n \in C[0,1] \cap C^2(0,1)$ with $\beta_n \geq \rho_n$ on $[0,1]$ such that $f(t,\beta_n(t)) + \beta_n''(t) \leq 0$ for $t \in (0,1)$; in addition, for each $t \in [0,1]$, we have that $\{\beta_n(t)\}$ is a nonincreasing sequence and $\lim_{n\to\infty} \beta_n(0) = \lim_{n\to\infty} \beta_n(1) = 0$* (2.35)

are satisfied. Then (2.1) has a solution in $C[0,1] \cap C^2(0,1)$.

PROOF. Fix $n \in \{3,4,\dots\}$. Consider $(2.30)^n$. The idea is to look at

$$\begin{gathered} y'' + f_n^{**}(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = y(1) = \rho_n, \end{gathered} \tag{2.36^n}$$

where

$$f_n^{**}(t,y) = \begin{cases} f\left(t, \beta_n(t)\right) + r\left(\beta_n(t) - y\right), & y > \beta_n(t), \\ f(t,y), & \rho_n \leq y \leq \beta_n(t), \\ f(t,\rho_n) + r(\rho_n - y), & y < \rho_n \text{ and } \dfrac{1}{n} \leq t \leq 1 - \dfrac{1}{n}, \\ f\left(\dfrac{1}{n}, \rho_n\right) + r(\rho_n - y), & y < \rho_n \text{ and } 0 \leq t \leq \dfrac{1}{n}, \\ f\left(1 - \dfrac{1}{n}, \rho_n\right) + r(\rho_n - y), & y < \rho_n \text{ and } 1 - \dfrac{1}{n} \leq t \leq 1, \end{cases}$$

and $r : \mathbf{R} \to [-1,1]$ is the radial retraction. Now Theorem 2.1(ii) implies that $(2.36)^n$ has a solution $y_n \in C[0,1] \cap C^2(0,1)$. Essentially, the same reasoning as in Theorem 2.3 yields

$$\rho_n \leq y_n(t) \leq \beta_n(t), \qquad \text{for } t \in [0,1],$$

hence y_n is a solution of $(2.30)^n$.

REMARK. Notice $y_n(t) \le \beta_3(t)$ for $t \in [0,1]$, since $\{\beta_n(t)\}$ is nonincreasing for each $t \in [0,1]$. Also as in Theorem 2.2, we have $y_n(t) \ge \alpha(t)$ for $t \in [0,1]$.

Look at the interval $[1/3, 2/3]$. Let

$$R_3 = \sup\left\{|f(x,y)| : x \in \left[\frac{1}{3},\frac{2}{3}\right] \text{ and } \alpha(x) \le y \le \beta_3(x)\right\}.$$

The mean value theorem implies that there exists $\tau \in (1/3, 2/3)$ with $|y_n'(\tau)| = 3|y_n(2/3) - y_n(1/3)| \le 6\sup_{[0,1]}\beta_3(t) \equiv L_3$. Hence, for $t \in [1/3, 2/3]$, we have

$$|y_n'(t)| \le |y_n'(\tau)| + \left|\int_\tau^t y_n''(x)\,dx\right| \le L_3 + R_3,$$

and so

$$\{y_n\}_{n=3}^{\infty} \text{ is a bounded, equicontinuous family on } \left[\frac{1}{3},\frac{2}{3}\right].$$

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N_3 of integers and a function $z_3 \in C[1/3, 2/3]$ with y_n converging uniformly to z_3 on $[1/3, 2/3]$ as $n \to \infty$ through N_3. Similarly,

$$\{y_n\}_{n=3}^{\infty} \text{ is a bounded, equicontinuous family on } \left[\frac{1}{4},\frac{3}{4}\right],$$

so there is a subsequence N_4 of N_3 and a function $z_4 \in C[1/4, 3/4]$ with y_n converging uniformly to z_4 on $[1/4, 3/4]$ as $n \to \infty$ through N_4. Note, $z_4 = z_3$ on $[1/3, 2/3]$ since $N_4 \subseteq N_3$. Proceed inductively to obtain subsequences of integers

$$N_3 \supseteq N_4 \supseteq \cdots \supseteq N_k \supseteq \cdots,$$

and functions

$$z_k \in C\left[\frac{1}{k}, 1 - \frac{1}{k}\right],$$

with

$$y_n \text{ converging uniformly to } z_k \text{ on } \left[\frac{1}{k}, 1-\frac{1}{k}\right] \text{ as } n \to \infty \text{ through } N_k$$

and

$$z_{k+1} = z_k \text{ on } \left[\frac{1}{k}, 1-\frac{1}{k}\right].$$

Define a function $y : [0,1] \to [0,\infty)$ by $y(x) = z_k(x)$ on $[1/k, 1-(1/k)]$ and $y(0) = y(1) = 0$. Notice y is well defined and $\alpha(t) \le y(t) \le \beta_3(t)$ for $t \in (0,1)$. Next fix $t \in (0,1)$. Then there exists $m \in \{3,4,5\dots\}$ with $t \in (1/m, 1-(1/m))$. Now y_n, $n \in N_m$, satisfies the integral equation

$$y_n(x) = y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{1/2}^{x} (s-x) f(s, y_n(s))\,ds, \qquad \text{for } x \in (0,1).$$

Let $n \to \infty$ through N_m to obtain as in Theorem 2.2 (here $r \in \mathbf{R}$ is as in Theorem 2.2),

$$z_m(x) = z_m\left(\frac{1}{2}\right) + r\left(x - \frac{1}{2}\right) + \int_{1/2}^{x} (s-x) f(s, z_m(s))\,ds, \qquad \text{for } x \in \left[\frac{1}{m}, 1-\frac{1}{m}\right],$$

i.e.,

$$y(x) = y\left(\frac{1}{2}\right) + r\left(x - \frac{1}{2}\right) + \int_{1/2}^{x} (s-x) f(s, y(s))\,ds, \qquad \text{for } x \in \left[\frac{1}{m}, 1-\frac{1}{m}\right].$$

Hence, $y \in C^2(0,1)$ with $y''(t) = -f(t, y(t))$ for each $t \in (0,1)$. It remains to show y is continuous at 0 and 1.

Let $\epsilon > 0$ be given. Now since $\lim_{n\to\infty} \beta_n(0) = 0$, there exists $n_0 \in \{3,4,\dots\}$ with $\beta_{n_0}(0) < \epsilon/2$. Since $\beta_{n_0} \in C[0,1]$, there exists $\delta_{n_0} > 0$ with

$$\beta_{n_0}(t) < \frac{\epsilon}{2}, \qquad \text{for } t \in [0, \delta_{n_0}].$$

Now for $n \geq n_0$ we have, since $\{\beta_n(t)\}$ is nonincreasing for each $t \in [0,1]$,

$$\beta_n(t) \leq \beta_{n_0}(t) < \frac{\epsilon}{2}, \qquad \text{for } t \in [0, \delta_{n_0}].$$

This together with the fact that $\alpha(t) \leq y_n(t) \leq \beta_n(t)$ for $t \in (0,1)$, implies that for $n \geq n_0$, we have

$$\alpha(t) \leq y_n(t) < \frac{\epsilon}{2}, \qquad \text{for } t \in [0, \delta_{n_0}].$$

Consequently,

$$0 \leq \alpha(t) \leq y(t) \leq \frac{\epsilon}{2} < \epsilon, \qquad \text{for } t \in [0, \delta_{n_0}],$$

and so y is continuous at 0. Similarly y is continuous at 1 and so $y \in C[0,1]$. ∎

We now discuss condition (2.14). Usually one can construct α explicitly from the differential equation; see [7,12]. For convenience, we now give a rather general result.

THEOREM 2.5. *Suppose (2.10), (2.11), and (2.15) hold. In addition, assume the following condition is satisfied:*

> *let $n \in \{3,4,\dots\}$ and associated with each n we have a constant ρ_n such that $\{\rho_n\}$ is a decreasing sequence with $\lim_{n\to\infty} \rho_n = 0$, and there exists a constant $k_0 > 0$ such that for $1/n \leq t \leq 1 - (1/n)$ and $0 < y \leq \rho_n$ we have $f(t,y) \geq k_0$.* (2.37)

Also, assume (2.13), here ρ_n is as in (2.37), holds. Then (2.1) has a solution in $C[0,1] \cap C^2(0,1)$.

PROOF. Clearly (2.37) implies that (2.12) holds. We now show that (2.14) is satisfied by explicitly constructing $\alpha(t)$; this is a standard construction off the sequence of constants $\{\rho_n\}$; see [10] for example.

The details are as follows. Let $0 \leq x \leq 1/3$ and

$$r_0(x) = \begin{cases} \rho_k \left(x - \dfrac{1}{k}\right) + \sum_{m=k+1}^{\infty} \rho_m \left(\dfrac{1}{m-1} - \dfrac{1}{m}\right), & x \in \left(\dfrac{1}{k}, \dfrac{1}{k-1}\right], \quad k = 4,5,\dots, \\ 0, & x = 0. \end{cases}$$

REMARKS.

(i) $\sum_{m=k+1}^{\infty} \rho_m (1/(m-1) - 1/m) \leq \rho_3 \sum_{m=4}^{\infty} (1/(m-1) - 1/m) = (1/3)\rho_3$.

(ii) Notice $r_0(x) = \int_0^x \phi(s)\,ds$ for $0 \leq x \leq 1/3$, where $\phi : [0,1/3] \to [0,\infty)$ is the step function defined by

$$\phi(t) = \begin{cases} 0, & t = 0, \\ \rho_k, & k \in \left(\dfrac{1}{k}, \dfrac{1}{k-1}\right], \quad k = 4,5,\dots. \end{cases}$$

Here $r_0 \in C[0,1/3]$, and notice $r_0(t) \leq \phi(t) \leq \rho_k$ for $t \in (1/k, 1/(k-1)]$, $k = 4,5,\dots$.

Next define

$$\psi(t) = \int_0^t \int_0^s r_0(x)\,dx\,ds, \qquad \text{for } 0 \leq t \leq \frac{1}{3}.$$

Notice $\psi(0) = 0$, $\psi > 0$ on $(0, 1/3]$, $\psi \in C^2[0, 1/3]$ and $\psi'' > 0$ for $t \in (0, 1/3)$. Also, $\psi(t) \le \rho_k$ for $t \in (1/k, 1/(k-1)]$, $k = 4, 5, \dots$, and so

$$f(t, y) \ge k_0, \qquad \text{for } (t, y) \in \left(0, \frac{1}{3}\right] \times \{y \in (0, \infty) : y \le \psi(t)\}. \tag{2.38}$$

Let

$$\alpha^\star(t) = \begin{cases} \psi(t), & 0 \le t < \dfrac{1}{3}, \\ q(t), & \dfrac{1}{3} \le t \le \dfrac{2}{3}, \\ \psi(1-t), & \dfrac{2}{3} < t \le 1. \end{cases}$$

Here $q : [1/3, 2/3] \to (0, \rho_3]$ is such that $q \in C^2[1/3, 2/3]$ with $q(1/3) = \psi(1/3) = q(2/3)$, $q'(1/3) = \psi'(1/3) = -q'(2/3)$, and $q''(1/3) = \psi''(1/3) = q''(2/3)$.

Notice since $0 < q(t) \le \rho_3$ for $t \in [1/3, 2/3]$, we have

$$f(t, y) \ge k_0, \qquad \text{for } (t, y) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \{y \in (0, \infty) : y \le q(t)\}. \tag{2.39}$$

Consequently, (2.38) and (2.39) imply

$$f(t, y) \ge k_0, \qquad \text{for } (t, y) \in (0, 1) \times \{y \in (0, \infty) : y \le \alpha^\star(t)\}. \tag{2.40}$$

Also, $\alpha^\star(0) = \alpha^\star(1) = 0$, $\alpha^\star > 0$ on $(0, 1)$, and $\alpha^\star \in C^2[0, 1]$. Finally, define

$$\alpha(t) = \eta\alpha^\star(t), \tag{2.41}$$

where

$$\eta = \min\left\{1, \frac{k_0}{\left|(\alpha^\star)''\right|_0 + 1}\right\}.$$

Now $\alpha \in C^2[0, 1]$ with $\alpha(0) = \alpha(1) = 0$ and $\alpha > 0$ on $(0, 1)$. Also since $\alpha(t) \le \alpha^\star(t)$, we have from (2.40) that

$$f(t, y) \ge k_0, \qquad \text{for } (t, y) \in (0, 1) \times \{y \in (0, \infty) : y < \alpha(t)\}.$$

In addition, for $(t, y) \in (0, 1) \times \{y \in (0, \infty) : y < \alpha(t)\}$, we have

$$f(t, y) + \alpha''(t) \ge k_0 + \alpha''(t) \ge k_0 - \frac{k_0 \left|(\alpha^\star)''(t)\right|}{\left|(\alpha^\star)''\right|_0 + 1} > k_0 - k_0 = 0.$$

Hence (2.14) is satisfied. Existence is now guaranteed from Theorem 2.2. ∎

Similarly, we have the following results; we use Theorems 2.3 and 2.4, respectively.

THEOREM 2.6. *Suppose (2.10), (2.11), (2.15), (2.37), and (2.29), here ρ_n is as in (2.37), hold. Then (2.1) has a solution in $C[0,1] \cap C^2(0,1)$.*

THEOREM 2.7. *Suppose (2.10), (2.34), (2.37), and (2.35), here ρ_n is as in (2.37), hold. Then (2.1) has a solution in $C[0,1] \cap C^2(0,1)$.*

EXAMPLE 2.1. The boundary value problem

$$\begin{gathered} y'' + \left(\frac{t}{y^\gamma} + \eta(t) y^\theta - \mu^2\right) = 0, \qquad 0 < t < 1, \\ y(0) = y(1) = 0, \quad \mu \ne 0, \qquad \gamma > 0, \quad 0 \le \theta < 1, \end{gathered} \tag{2.42}$$

has a solution; here $\eta \in C(0, 1)$, $\eta > 0$ on $(0, 1)$ and $\int_0^1 \eta(t)\, dt < \infty$.

We will apply Theorem 2.5. Notice first that (2.37), with

$$\rho_n = \left(\frac{1}{n(\mu^2+1)}\right)^{1/\gamma} \quad \text{and} \quad k_0 = 1$$

is true, since for $1/n \le t \le 1 - 1/n$ and $0 < y \le \rho_n$, we have

$$f(t,y) \ge \frac{t}{y^\gamma} - \mu^2 \ge \frac{1}{ny^\gamma} - \mu^2 \ge (\mu^2+1) - \mu^2 = 1.$$

Also with $q_1(t) = t$, $g(y) = y^{-\gamma}$, $q_2(t) = \eta(t) + \mu^2$, and $h(y) = y^\theta + 1$, it is easy to check that (2.11),(2.13) since $0 \le \theta < 1$, and (2.15) hold. Existence of a solution to (2.42) is now guaranteed from Theorem 2.5.

EXAMPLE 2.2. The boundary value problem

$$\begin{aligned} y'' + \left(\frac{t^2}{4y^2} - \mu^2\right) &= 0, \qquad 0<t<1, \\ y(0) = y(1) &= 0, \qquad \mu \ne 0, \end{aligned} \tag{2.43}$$

has a solution.

We will apply Theorem 2.6. Notice (2.37), with

$$\rho_n = \left(\frac{1}{4n^2(\mu^2+1)}\right)^{1/2} \quad \text{and} \quad k_0 = 1$$

is true. Also let $q_1(t) = t^2/4$, $g(y) = y^{-2}$, $q_2(t) = \mu^2$, and $h(y) = 1$. Clearly, (2.11) and (2.15) hold. Finally, (2.29) is satisfied with

$$\beta(t) = \frac{t}{2\mu} + \rho_3.$$

Existence of a solution to (2.43) is now guaranteed from Theorem 2.6.

EXAMPLE 2.3. The boundary value problem

$$\begin{aligned} y'' + \left(\frac{t^2(1-t)^2}{4y^2} + \frac{y^2}{4} - \mu^2\right) &= 0, \qquad 0<t<1, \\ y(0) = y(1) &= 0, \qquad \mu \ne 0, \end{aligned} \tag{2.44}$$

has a solution.

We will apply Theorem 2.7. Notice (2.37), with

$$\rho_n = \left(\frac{1}{4n^4(\mu^2+1)}\right)^{1/2} \quad \text{and} \quad k_0 = 1$$

is true, since for $1/n \le t \le 1 - 1/n$ and $0 < y \le \rho_n$, we have

$$f(t,y) \ge \frac{1}{4n^4\rho_n^2} - \mu^2 = 1.$$

Also with $q_1(t) = (t^2(1-t)^2)/4$, $g(y) = y^{-\gamma}, q_2(t) = 1$, and $h(y) = y^2/4 + \mu^2$, we have that (2.34) is satisfied. Finally, (2.35) with $\beta_n(t) = t(1-t) + \rho_n$ holds since

$$f(t, \beta_n(t)) + \beta_n''(t) \le \frac{1}{4} + \frac{(1+\rho_n)^2}{4} - \mu^2 - 2 \le \frac{1}{4} + 1 - \mu^2 - 2 < 0.$$

Existence of a solution to (2.44) is now guaranteed from Theorem 2.7.

3. NONLINEAR BOUNDARY CONDITION

In this section, we discuss the singular problem

$$\begin{gathered} y'' + f(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = 0, \qquad \theta\left(y'(1)\right) + y(1) = 0, \end{gathered} \tag{3.1}$$

where θ may be nonlinear.

THEOREM 3.1. *Suppose (2.10)–(2.12) and (2.15) hold. In addition, suppose the following conditions are satisfied:*

$$\theta : \mathbf{R} \to \mathbf{R} \text{ is continuous and nondecreasing with } \theta(0) = 0; \tag{3.2}$$

there exists a constant $M > 0$ *such that for* $z > 0$, $\int_0^z (du/g(u)) \le \int_0^1 xq_1(x)\,dx + (h(z)/g(z)) \int_0^1 xq_2(x)\,dx + \int_0^{\rho_3} (du/g(u))$ *implies* $z \le M$; (3.3)

there exists a function $\alpha \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ *with* $\alpha(0) = \theta(\alpha'(1)) + \alpha(1) = 0$, $\alpha > 0$ *on* $(0,1)$ *such that* $f(t,y) + \alpha''(t) > 0$ *for* $(t,y) \in (0,1) \times \{y \in (0,\infty) : y < \alpha(t)\}$; (3.4)

and

$$\int_{1/2}^1 q_i(x) g\left(\alpha(x)\right)\,dx < \infty, \qquad i = 1,2. \tag{3.5}$$

Then (3.1) has a solution in $C[0,1] \cap C^1(0,1] \cap C^2(0,1)$.

PROOF. Fix $n \in \{3,4,\dots\}$. We first show

$$\begin{gathered} y'' + f(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = \rho_n, \qquad \theta\left(y'(1)\right) + y(1) = \rho_n, \end{gathered} \tag{3.6^n}$$

has a $C^1[0,1] \cap C^2(0,1)$ solution. The idea is to look at

$$\begin{gathered} y'' + f^\star(t,y) = 0, \qquad 0 < t < 1, \\ y(0) = \rho_n, \qquad \theta\left(y'(1)\right) + y(1) = \rho_n. \end{gathered} \tag{3.7^n}$$

We establish that $(3.7)^n$ has a $C^1[0,1] \cap C^2(0,1)$ solution; here $f^\star$ is as in Theorem 2.2.

Look at the family of problems

$$\begin{gathered} y'' + \lambda f^\star(t,y) = 0, \qquad 0 < t < 1, \quad 0 < \lambda < 1, \\ y(0) = \rho_n, \qquad \lambda\theta\left(y'(1)\right) + y(1) = \rho_n. \end{gathered} \tag{3.8$^n_\lambda$}$$

We claim

$$y(t) \ge \rho_n, \qquad t \in [0,1], \tag{3.9}$$

for any solution $y \in C^1[0,1] \cap C^2(0,1)$ to $(3.8)^n_\lambda$. To see this, suppose $y - \rho_n$ has a negative minimum at $t_0 \in (0,1]$. If $t_0 \in (0,1)$, then we obtain a contradiction as in Theorem 2.2. It remains to consider the case $t_0 = 1$. Then $y'(1) \le 0$, and so $\theta(y'(1)) \le 0$ from (3.2). However,

$$\lambda\theta\left(y'(1)\right) = \rho_n - y(1) > 0,$$

a contradiction. Thus (3.9) holds.

Suppose the absolute maximum of y occurs at $t_n \in [0,1]$. In fact, we may take $t_n \in (0,1)$ and so $y'(t_n) = 0$. To see this notice if $y(t_n) = \rho_n$, then $y \equiv \rho_n$. Next if $y(t_n) > \rho_n$, then if $t_n = 1$ we have $y'(1) \geq 0$, and so

$$0 \leq \lambda\theta\left(y'(1)\right) = \rho_n - y(1) < 0,$$

a contradiction.

For $x \in (0,1)$, we have

$$\frac{-y''(x)}{g\left(y(x)\right)} \leq q_1(x) + q_2(x)\frac{h\left(y(x)\right)}{g\left(y(x)\right)}, \tag{3.10}$$

and integrate from $t\,(t < t_n)$ to t_n and then from 0 to t_n to obtain (as in Theorem 2.2),

$$\int_0^{y(t_n)} \frac{du}{g(u)} \leq \int_0^1 xq_1(x)\,dx + \frac{h(y(t_n))}{g(y(t_n))}\int_0^1 xq_2(x)\,dx + \int_0^{\rho_3} \frac{du}{g(u)}. \tag{3.11}$$

Now (3.3) implies $y(t_n) \leq M$, and so

$$\rho_n \leq y(t) \leq M, \qquad \text{for } t \in [0,1]. \tag{3.12}$$

Also the mean value theorem implies that there exists $\tau \in (0,1)$ with $|y'(\tau)| = |y(1)-y(0)| \leq 2M$. For $t \in [0,1]$, we have

$$\begin{aligned} |y'(t)| &\leq |y'(\tau)| + \left|\int_\tau^t |f^\star\left(x,y(x)\right)\,dx\right| \\ &\leq 2M + g(\rho_n)\int_0^1 \left[q_1(x) + q_2(x)\frac{h(M)}{g(M)}\right]\,dx \equiv M_1. \end{aligned}$$

Define the mappings

$$L,\ F : C^1_{\rho_n}[0,1] \to C_0[0,1] \times \mathbf{R},$$

by

$$Ly(t) = \left(y'(t) - y'(0),\ \rho_n - y(1)\right) \quad \text{and} \quad Fy(t) = \left(-\int_0^t f^\star\left(x,y(x)\right)\,dx,\ \theta\left(y'(1)\right)\right).$$

Here $C_0[0,1] = \{u \in C[0,1] : u(0) = 0\}$ and $C^1_{\rho_n}[0,1] = \{u \in C^1[0,1] : u(0) = \rho_n\}$. Now F is completely continuous. Also, if $Ly = (u(t),\gamma)$, then

$$y(t) = \rho_n - t\left(\gamma + \int_0^1 u(x)\,dx\right) + \int_0^t u(x)\,dx,$$

hence, L^{-1} exists and is continuous.

Solving $(3.8)^n_\lambda$ is equivalent to finding a fixed point of $y = \lambda L^{-1}Fy = \lambda Ny$, where $N = L^{-1}F : C^1_{\rho_n}[0,1] \to C^1_{\rho_n}[0,1]$ is completely continuous. Let

$$U = \left\{u \in C^1_{\rho_n}[0,1] : |u|_1 < \max\left\{M, M_1\right\} + 1\right\}, \quad K = C^1_{\rho_n}[0,1], \quad \text{and} \quad E = C^1[0,1];$$

here $|u|_1 = \max\{|u|_0, |u'|_0\}$. Now Theorem 1.2 implies that $(3.7)^n$ has a solution $y_n \in C^1[0,1] \cap C^2(0,1)$. Also $\rho_n \leq y_n(t) \leq M$ for $t \in [0,1]$, so y_n is a solution of $(3.6)^n$.

Next we show

$$\alpha(t) \leq y_n(t) \leq M, \qquad \text{for } t \in [0,1]. \tag{3.13}$$

If this is not true then $y_n - \alpha$ would have a negative minimum at, say, $t_0 \in (0,1]$. If $t_0 \in (0,1)$, then we obtain a contradiction as in Theorem 2.2. If $t_0 = 1$, then $(y_n - \alpha)'(1) \leq 0$, i.e., $y_n'(1) \leq \alpha'(1)$, and $0 < y_n(1) < \alpha(1)$. However,

$$0 < \rho_n + \left[\alpha(1) - y_n(1)\right] = \theta\left(y_n'(1)\right) - \theta\left(\alpha'(1)\right) \leq 0,$$

a contradiction. Hence, (3.13) is true.

We have

$$\frac{-y_n''(x)}{g\left(y_n(x)\right)} \le q_1(x) + q_2(x)\frac{h(M)}{g(M)}, \qquad \text{for } x \in (0,1).$$

Integrate from 0 to 1 to obtain

$$\frac{-y_n'(1)}{g\left(y_n(1)\right)} + \frac{y_n'(0)}{g(\rho_n)} + \int_0^1 \left\{ \frac{-g'\left(y_n(x)\right)}{g^2\left(y_n(x)\right)} \right\} \left[y_n'(x)\right]^2 \, dx \le \int_0^1 \left[q_1(x) + q_2(x)\frac{h(M)}{g(M)} \right] dx.$$

Then since $y_n'(0) \ge 0$ and $y_n'(1) \le 0$ (notice $\theta(y_n'(1)) = \rho_n - y_n(1) \le 0$ implies $y_n'(1) \le 0$), we have

$$\int_0^1 \left\{ \frac{-g'\left(y_n(x)\right)}{g^2\left(y_n(x)\right)} \right\} \left[y_n'(x)\right]^2 \, dx \le \int_0^1 \left[q_1(x) + q_2(x)\frac{h(M)}{g(M)} \right] dx \equiv M_2. \tag{3.14}$$

Now consider

$$I(z) = \int_0^z \frac{\left[-g'(u)\right]^{1/2}}{g(u)} \, du,$$

and notice I (as in Theorem 2.2) for $t, s \in [0,1]$ that

$$\left|I(y_n(t)) - I(y_n(s))\right| \le M_2^{1/2} |t-s|^{1/2}.$$

Consequently,

$$\{y_n\}_{n=3}^{\infty} \text{ is a bounded, equicontinuous family on } [0,1]. \tag{3.15}$$

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N of integers and a function $y \in C[0,1]$ with y_n converging uniformly on $[0,1]$ to y as $n \to \infty$ through N. Also, $y(0) = 0$ and $\alpha(t) \le y(t) \le M$ for $t \in [0,1]$. Now $y_n, n \in N$, satisfies the integral equation

$$\begin{aligned} y_n(t) = y_n\left(\frac{1}{2}\right) + \left(t - \frac{1}{2}\right)\Omega^{-1}\left(\rho_n - y_n\left(\frac{1}{2}\right) - \int_{1/2}^1 \left(x - \frac{1}{2}\right) f\left(x, y_n(x)\right) dx\right) \\ + \int_{1/2}^t \left(x - \frac{1}{2}\right) f\left(x, y_n(x)\right) dx + \int_t^1 \left(t - \frac{1}{2}\right) f\left(x, y_n(x)\right) dx, \end{aligned} \tag{3.16}$$

for $t \in [0,1]$. Here $\Omega : \mathbf{R} \to \mathbf{R}$ is given by $\Omega(x) = x/2 + \theta(x)$; notice Ω is strictly increasing. We would like to let $n \to \infty$ through N in (3.16). First, notice

$$\int_{1/2}^1 \left|f\left(x, y_n(x)\right)\right| dx \le \int_{1/2}^1 g\left(\alpha(x)\right) \left\{ q_1(x) + q_2(x)\frac{h(M)}{g(M)} \right\} dx < \infty.$$

Fix $t \in (0,1]$. Let $n \to \infty$ through N in (3.16), and so the Lebesgue dominated convergence theorem implies

$$\begin{aligned} y(t) = y\left(\frac{1}{2}\right) + \left(t - \frac{1}{2}\right)\Omega^{-1}\left(-y\left(\frac{1}{2}\right) - \int_{1/2}^1 \left(x - \frac{1}{2}\right) f\left(x, y(x)\right) dx\right) \\ + \int_{1/2}^t \left(x - \frac{1}{2}\right) f\left(x, y(x)\right) dx + \int_t^1 \left(t - \frac{1}{2}\right) f\left(x, y(x)\right) dx, \end{aligned}$$

for $t \in (0,1]$. Also, for $t \in (0,1]$, we have

$$y'(t) = \Omega^{-1}\left(-y\left(\frac{1}{2}\right) - \int_{1/2}^1 \left(x - \frac{1}{2}\right) f\left(x, y(x)\right) dx\right) + \int_t^1 f\left(x, y(x)\right) dx,$$

and so $y' \in C(0,1]$. In addition, $y''(t) = -f(t,y(t))$ for $t \in (0,1)$, and hence, $y \in C^2(0,1)$. Finally,

$$\begin{aligned}
\theta\left(y'(1)\right) + y(1) &= \theta\left(\Omega^{-1}\left(-y\left(\frac{1}{2}\right) - \int_{1/2}^{1}\left(x - \frac{1}{2}\right) f\left(x, y(x)\right)\, dx\right)\right) \\
&\quad + \frac{1}{2}\Omega^{-1}\left(-y\left(\frac{1}{2}\right) - \int_{1/2}^{1}\left(x - \frac{1}{2}\right) f\left(x, y(x)\right)\, dx\right) \\
&= y\left(\frac{1}{2}\right) + \int_{1/2}^{t}\left(x - \frac{1}{2}\right) f\left(x, y(x)\right)\, dx = 0.
\end{aligned}$$

∎

Now suppose (2.37) is satisfied. Let $\alpha^\star$ and α be as in Theorem 2.5. Notice $\alpha^\star(1) = 0$ and $(\alpha^\star)'(t) = -\int_0^{1-t} r_0(x)\, dx$ for $2/3 \le t \le 1$. Consequently, $(\alpha^\star)'(1) = 0$ and so $\theta(\alpha'(1)) + \alpha(1) = \theta(0) + 0 = 0$. However, since (3.5) must be satisfied it is desirable to construct the "best" α. Usually it is possible to obtain an explicit α from the differential equation. We conclude by giving a general result for the boundary value problem

$$\begin{aligned}
y'' + f(t,y) &= 0, \qquad 0 < t < 1, \\
y(0) &= 0, \\
ky'(1) + y(1) &= 0, \qquad k \ge 0 \text{ a constant.}
\end{aligned} \tag{3.17}$$

REMARK. Note (3.17) is a special case of (3.1); here $\theta(u) = ku$.

THEOREM 3.2. *Suppose (2.10), (2.11), (2.37), and (3.3) are satisfied. In addition, assume*

$$\int_{1/2}^{1} q_i(x) g\left(a_0(1-x)\right)\, dx < \infty, \qquad i = 1,2 \text{ for any } a_0 > 0, \text{ if } k = 0, \tag{3.18}$$

and

$$\text{there exists } \tau \in (0,1) \text{ with } f(t,y) > 0, \qquad \text{for } t \in [\tau, 1) \text{ and } 0 < y \le \rho_3, \tag{3.19}$$

hold. Then (3.17) has a solution in $C[0,1] \cap C^1(0,1] \cap C^2(0,1)$.

PROOF. Let γ denote the α constructed in Theorem 2.5. Without loss of generality, assume $1/3 < \tau \le 2/3$. Define

$$\mu(t) = \begin{cases} \gamma(t), & 0 \le t \le \dfrac{1}{3}, \\ \omega(t), & \dfrac{1}{3} < t < \tau, \\ \dfrac{k\gamma(\tau)}{(k+1) - \tau} + \dfrac{\gamma(\tau)(1-t)}{(k+1) - \tau}, & t \ge \tau, \end{cases}$$

and $\alpha(t) = \eta\mu(t)$, where

$$\eta = \min\left\{1, \frac{k_0}{|\mu''|_0 + 1}\right\}.$$

Here $\omega : [1/3, \tau] \to (0, \rho_3]$ is such that $\omega \in C^2[1/3, \tau]$ with $\omega(1/3) = \gamma(1/3)$, $\omega'(1/3) = \gamma'(1/3)$, $\omega''(1/3) = \gamma''(1/3)$, $\omega(\tau) = \gamma(\tau)$, $\omega'(\tau) = (-\gamma(\tau))/((k+1) - \tau)$, and $\omega''(\tau) = 0$.

We now claim that α satisfies (3.4). First notice $\alpha(0) = k\alpha'(1) + \alpha(1) = 0$. We also have as in Theorem 2.5 that

$$f(t,y) + \alpha''(t) > 0, \qquad \text{for } (t,y) \in (0,\tau) \times \{y \in (0,\infty) : y < \alpha(t)\}. \tag{3.20}$$

Now for $t > \tau$, we have $\alpha''(t) = 0$ and also

$$\alpha(t) \le \mu(t) \le \frac{k\rho_3}{(k+1)-\tau} + \frac{(1-\tau)\rho_3}{(k+1)-\tau} = \rho_3.$$

Consequently,

$$f(t,y) + \alpha''(t) = f(t,y) > 0, \qquad \text{for } (t,y) \in [\tau,1) \times \{y \in (0,\infty) : y < \alpha(t)\}. \tag{3.21}$$

Thus, (3.20) and (3.21) imply that α satisfies (3.4). Notice also that (3.5) is satisfied with the above α since (3.18) holds. Existence of a $C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ solution to (3.17) is now guaranteed from Theorem 3.1. ∎

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